Reply to "Comment on 'Properties of the massive Thirring model from the *XYZ* spin chain' "

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We elaborate in more detail why the lattice calculation by Kolanovic *et al.*, Phys. Rev. D **62**, 025021 (2000), was done correctly and argue that increasing the number of sites is not expected to change our conclusions about the mass spectrum.

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In the Comment on our paper [1] Fujita, Kobayashi, and Takahashi claim that our results for the mass spectrum of the massive Thirring model (MTM) are not reliable. In particular, they claim that if one uses spin chain regularization it is *necessary* to diagonalize the spin chain Hamiltonian with a number of sites N larger than 1000. We now explain why we disagree with this criticism.

The criticism of Fujita *et al.* is based on the following argument. If one makes the standard spin chain regularization of the MTM (which is the XYZ spin 1/2 chain [2,3]) to obtain some reasonable results on continuum extrapolation, one has to satisfy the condition

$$\frac{2\pi}{N} \ll am_0 \ll 2\pi. \tag{1}$$

Here N is the number of sites, a is the lattice spacing, and m_0 is a bare mass parameter. Using Eq. (1) Fujita *et al.* claim that if one wants to obtain any reliable information on the bound state of the MTM one has to take $N>1000$, which is much larger than our $N\leqslant 16$. Moreover, they claim that for values of the parameters used in [1] the left inequality in Eq. (1) is even completely violated, i.e., $2\pi/N > am_0$. In addition they claim that the mass gap in our calculation is approximately equal to the "resolution" $2\pi/L$. From all this, Fujita *et al.* conclude that our results [1] cannot be very reliable for the bound state spectrum of the massive Thirring model.

Let us now review standard lattice philosophy. The continuum regime in lattice calculations is obtained when the correlation length ξ is much larger than the lattice spacing a and at the same time much smaller than the spatial extension $L=Na$, i.e.,

$$a \ll \xi \ll Na. \tag{2}$$

As $\xi = 1/M$, where M is the the mass gap (the mass of the lightest particle), Eq. (2) can be equivalently written as

$$\frac{1}{N} \ll Ma \ll 1. \tag{3}$$

Now, hardware limitations make restrictions on N. For example, in lattice (quenched) QCD, the maximum lattices that are presently calculable have $N\leqslant 64$ (see, e.g., [4]). More importantly here, for exact diagonalization in two dimensions $N<30$. It follows that "$\ll$" in Eq. (3) at best means 5–8 times smaller. So in practical calculations one effectively imposes the condition

$$\frac{1}{N} < Ma < 1, \tag{4}$$

and from the quality of scaling and the accuracy of the continuum extrapolation (usually using different methods as a check) one decides whether the extrapolated results are a good approximation of the continuum theory. Of course, one cannot exclude the possibility that for values of the scaling parameter larger than those accessed there is a complete change of scaling behavior so that the obtained extrapolated results are wrong.[1]

Let us now apply the above textbook analysis to our lattice calculation (XYZ spin chain regularization of the MTM) [1]. For the sake of clarity we restrict ourselves to an interval of the coupling constant where the elementary fermion is the particle with the lowest mass. First, it is easy to see that Eq. (3) is *not* equal to Eq. (1) used by Fujita *et al.* It was shown in [3] that (using notation from [1])

$$m_0 a = \frac{8\gamma}{\pi} \sin\gamma \left(\frac{Ma}{4}\right)^{2\gamma/\pi}.$$

In particular, let us analyze the left inequality in Eq. (4). In our extrapolation we had $Ma>0.2$, which for $N=16$ is reasonably larger than $1/N=0.06$. It follows that in our analysis the continuum condition is fairly well satisfied when the *proper* condition is used, contrary to the claim of Fujita *et al.*

As we mentioned above, there is always the possibility that for larger N something dramatic happens with the scaling law and that our extrapolations made with $N\leqslant 16$ are

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[1] For a nice discussion see Sec. 9.5 in Ref. [5].

incorrect. There are several reasons why we believe that this should not be expected in the case of the XYZ chain.

(1) The global properties of the energy spectrum are as expected for quantum field theory with a calculated mass spectrum. In particular, there are states with energies corresponding to two-fermion and two-(first-)breather states.

(2) The same technique was previously successfully applied for similar perturbed conformal field theories, e.g., the ordinary and tricritical Ising models in a magnetic field [6]. It was shown there that passing from $N \leqslant 14$ to $N \leqslant 21$ only slightly improved the precision, and the conclusions about the spectra remained the same. In fact, it was shown that very accurate results can already be obtained for lattices with $N \leqslant 24$ [7].

(3) Our results agree with the Dashen-Haslacher-Neveu (DHN) mass formula [8] obtained by a number of different methods. It is very hard to imagine how a bad extrapolation can agree with an exact analytic result. We should also mention that a similar analysis for lattice regularization of the sine-Gordon model (a periodic XXZ chain in a transverse magnetic field) also gave results consistent with the DHN formula [9].

(4) In addition to the mass ratios in the $L \to \infty$ limit we obtained the anomalous dimensions of the corresponding states; in particular, we obtained for the first time the calculated dimension of the second breather. Our result was subsequently confirmed by an analytical calculation [10].

We conclude that our numerical analysis made in [1] was done correctly and that all results should be trusted, in the sense that enlarging the lattice would just increase the numerical precision and leave all conclusions unchanged.

Finally, we would like to comment on the Bethe ansatz solution of the MTM. The spectrum of the MTM can be found by solving the Bethe ansatz equations in the continuum approximation [11]. The spectrum found there agrees with the results of [8]. Recently, it was argued [12] that there exist no complex solutions to the Bethe equations and that there is only one bound state. In [11] it is precisely the complex solutions (so-called Bethe strings) that describe bound states of the MTM. We performed a numerical analysis of the Bethe equations for the MTM and tried to reproduce the result of [12]. Our results indicate that the powers α that determine the scaling behavior of the energies of the few lowest states with the density ρ differ and even vary with ρ (for the definitions see Sec. 4 in [12]). Therefore we were unable to extract conclusive results by letting $\rho \to \infty$. An indication that numerical iterations might easily miss the complex solutions (that nevertheless exist) comes from our study of the Bethe ansatz equations for different spin chains [13]. There we studied how the complex (string) solution in the two- and three-particle sectors emerges and disappears when one changes the parameters of the model. We found that iterative numerical methods fail to converge on string solutions, although they exist and can be found analytically.

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